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The space of m-ary differential operators as a module over the Lie algebra of vector fields

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Abstract

The space $\mathcal{D}_{\underline{\lambda};\mu}$, where $\underline{\lambda} = (\lambda_1, \dots, \lambda_m)$, of m-ary differential operators acting on weighted densities is a $(m+1)$-parameter family of modules over the Lie algebra of vector fields. For almost all the parameters, we construct a canonical isomorphism between the space $\mathcal{D}_{\underline{\lambda};\mu}$ and the corresponding space of symbols as $\mathfrak{sl}(2)$-modules. This yields to the notion of the $\mathfrak{sl}(2)$-equivariant symbol calculus for m-ary differential operators. We show, however, that these two modules cannot be isomorphic as $\mathfrak{sl}(2)$-modules for some particular values of the parameters. Furthermore, we use the symbol map to show that all modules $\mathcal{D}^2_{\underline{\lambda};\mu}$ (i.e., the space of second-order operators) are isomorphic to each other, except for a few modules called singular.

Keywords: Equivariant quantization; Invariant operators; Module of differential operators

1. Introduction

In this paper let M be either $\mathbb{R}$ or S^1; let $\mathcal{F}_\lambda$ be the space of weighted densities on M of weight λ, i.e., the space of sections of the line bundle $(T^*M)^{\otimes\lambda}$, where $\lambda \in \mathbb{R}$. This space has the following structure as a Vect(M)-module: for any $a\,(dx)^\lambda \in \mathcal{F}_\lambda$ and $X \in \mathrm{Vect}(M)$, we put

$$L_X^\lambda(a\,(dx)^\lambda) = (X(a) + \lambda\, a\, \mathrm{div}\, X)\,(dx)^\lambda. \tag{1.1}$$

Denote by $\mathcal{D}_{\underline{\lambda};\mu}$, where $\underline{\lambda} = (\lambda_1, \dots, \lambda_m)$, the space of m-ary linear differential operators:

$$\mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_m} \to \mathcal{F}_\mu.$$

The action by the Lie derivative on $\mathcal{D}_{\underline{\lambda};\mu}$ defines a module structure over the Lie algebra of vector fields, Vect(M). The space of unary differential operators viewed as a Vect(M)-module is a classical object (see, e.g., [25]).

In this paper, we study the geometry of the modules $\mathcal{D}_{\underline{\lambda};\mu}$ understood in the sense of Klein: as Lie group actions (or Lie algebra actions) on a manifold. We will be dealing with the Lie algebra of smooth vector fields, Vect(M), and $\mathfrak{sl}(2)$ embedded into Vect(M) via infinitesimal projective transformations:

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$$\mathfrak{sl}(2) \simeq \mathrm{Span}\{\partial_x, x\,\partial_x, x^2\partial_x\}. \tag{1.2}$$

The quotient module $\mathcal{D}^k_{\underline{\lambda};\mu}/\mathcal{D}^{k-1}_{\underline{\lambda};\mu}$ can be decomposed into $\binom{k+m-1}{m-1}$ components that transform under coordinate changes as $(\delta - k)$-densities, where $\delta = \mu - \sum_{j=1}^m \lambda_j$. Therefore, the multiplication of these components by any non-singular matrix, say α, gives rise to an isomorphism

$$\sigma^\alpha : \mathcal{D}^k_{\underline{\lambda};\mu}/\mathcal{D}^{k-1}_{\underline{\lambda};\mu} \xrightarrow{\simeq} \mathcal{F}_{\delta-k} \oplus \cdots \oplus \mathcal{F}_{\delta-k}.$$

The map σ^α is what we call the *principal symbol*. By the very definition, the principal symbol is Vect(M)-equivariant but not unique if $m > 1$. Let us consider the graded space

$$\mathcal{S}_\delta = \bigoplus_{k\geq 0} \mathcal{D}^k_{\underline{\lambda};\mu}/\mathcal{D}^{k-1}_{\underline{\lambda};\mu}$$

associated with the natural filtration of $\mathcal{D}_{\underline{\lambda};\mu}$. A *symbol* map is a linear bijection

$$\sigma^\alpha_{\underline{\lambda},\mu} : \mathcal{D}_{\underline{\lambda};\mu} \longrightarrow \mathcal{S}_\delta$$

such that the highest-order term of $\sigma^\alpha_{\underline{\lambda},\mu}(A)$, where $A \in \mathcal{D}_{\underline{\lambda};\mu}$, coincides with the principal symbol $\sigma^\alpha(A)$. In the unary case, equivariant symbol calculus has been first introduced in [7], then studied in [14]. A generalization to multi-dimensional manifolds has been first studied in [10,21], then studied in [1,2,6,17,22,23]. In the binary case, the existence of the symbol map has been investigated in [5] and an explicit formula has been given for the space of second-order operators. We show, for almost all $\underline{\lambda}$ and μ, that there exists a $\mathfrak{sl}(2)$-equivariant symbol map for every m. As in the unary case, we show that the symbol maps are not Vect(M)-equivariant except for $k = 1$ or $k = 2$ but only for particular values of $\underline{\lambda}$ and μ. Although the symbol map is not unique for $m > 1$ – unlike the unary case – the uniqueness can be understood as follows: once the symbol map σ^α is fixed, the corresponding $\mathfrak{sl}(2)$-equivariant map $\sigma^\alpha_{\underline{\lambda},\mu}$ is unique.

Furthermore, we investigate for which parameters $(\underline{\lambda}, \mu)$ and $(\underline{\rho}, \eta)$, we have $\mathcal{D}^k_{\underline{\lambda};\mu} \simeq \mathcal{D}^k_{\underline{\rho};\eta}$ as Vect(M)-modules. We only deal with $k = 1, 2$. In the unary case, the classification problem of such modules has first been raised and studied in [11]; and comprehensive results were obtained in the papers [11,14,20]. As for the unary case, a necessary condition for the isomorphism is imposed by

$$\delta := \mu - \sum_{j=1}^m \lambda_j = \eta - \sum_{j=1}^m \rho_j.$$

We prove that all modules are isomorphic to each other, provided they have the same δ, except:

- the module $\mathcal{D}^1_{0,\ldots,0;\delta}$;
- the modules $\mathcal{D}^2_{0,\ldots,0;\delta} \simeq \mathcal{D}^2_{1-\delta,\ldots,0;1} \simeq \cdots \simeq \mathcal{D}^2_{0,\ldots,1-\delta;1}$.

In the generic case, we use the symbol map $\sigma^\alpha_{\underline{\lambda},\mu}$ to build-up the isomorphism between $\mathcal{D}^k_{\underline{\lambda};\mu}$ and $\mathcal{D}^k_{\underline{\rho};\eta}$. However, for some values of δ, called *resonant*, we proceed by a direct computation. The proof in all cases is based on the locality of any Vect(M)-isomorphism.

2. The space of m-ary differential operators as a Vect(M)-module

We fix a natural number m. In order to avoid clutter, we have found that it is convenient to use the following notations:

- Denote by $\underline{i}$ either the m-tuple $(i_1, \ldots, i_m)$ or the indices $i_1, \ldots, i_m$, as, for instance, $a_{\underline{i}} = a_{i_1,\ldots,i_m}$. The difference should be discernable from the context.
- Denote by $|\underline{i}|$ the sum $\sum_{j=1}^m i_j$.
- Denote by $[\tau]_p$ the square matrix $[\tau^{\underline{i}}_{\underline{j}}]$ of size $\binom{p+m-1}{m-1} \times \binom{p+m-1}{m-1}$ for $|\underline{i}| = |\underline{j}| = p$.
- Denote $\mathbf{1}_i := (0, \ldots, 0, 1, 0 \ldots, 0)$, where 1 is in the i-th position.

- Denote by $\mathcal{S}^{(i)}_\lambda = \bigoplus \mathcal{F}_\lambda$, where $\mathcal{F}_\lambda$ is counted $\binom{i+m-1}{m-1}$ times.

Consider m-ary differential operators that act on weighted densities:

$$A : \mathcal{F}_{\lambda_1} \otimes \cdots \otimes \mathcal{F}_{\lambda_m} \to \mathcal{F}_\mu \qquad \underline{\varphi} \mapsto \sum_{r\geq 0} \sum_{|\underline{i}|=r} a_{\underline{i}}\, \partial_{\underline{i}}\, \underline{\varphi}, \tag{2.1}$$

where $a_{\underline{i}}$ are smooth functions on M. We denote by $\mathcal{D}^k_{\underline{\lambda};\mu}$, where $\underline{\lambda} = (\lambda_1, \ldots, \lambda_m)$, the space of kth-order m-ary differential operators (2.1) endowed with the following $\mathrm{Vect}(M)$-module structure.

For all $X \in \mathrm{Vect}(M)$, we define ($L^{\lambda_j}_X$ is the action (1.1)):

$$L^{\underline{\lambda};\mu}_X(A) = L^\mu_X \circ A - \sum_{j=1}^m A(\ldots, L^{\lambda_j}_X(-), \ldots). \tag{2.2}$$

Let us denote by $a^X_{\underline{i}}$ the coefficients of the operator $L^{\underline{\lambda};\mu}_X(A)$. We have

Proposition 2.1. *The coefficients $a^X_{\underline{i}}$ can be expressed in terms of the coefficients $a_{\underline{i}}$ as follows (where $\delta = \mu - |\underline{\lambda}|$):*

$$\begin{aligned} a^X_{\underline{s}} \;&=\; L^{\delta-|\underline{s}|}_X a_{\underline{s}} - \sum_{i\geq s_1+1}^k \binom{i}{i+1-s_1} X^{(i+1-s_1)}\, a_{\underline{s}_{|s_1=i}} - \cdots - \sum_{i\geq s_m+1}^k \binom{i}{i+1-s_m} X^{(i+1-s_m)}\, a_{\underline{s}_{|s_m=i}} \\ &\quad - \lambda_1 \sum_{i\geq s_1+1}^k \binom{i}{i-s_1} X^{(i+1-s_1)}\, a_{\underline{s}_{|s_1=i}} - \cdots - \lambda_m \sum_{i\geq s_m+1}^k \binom{i}{i-s_m} X^{(i+1-s_m)}\, a_{\underline{s}_{|s_m=i}}, \\ a^X_{\underline{0}} &= L^\delta_X a_{\underline{0}} - \lambda_1 \sum_{i=1}^k X^{i+1}\, a_{i\mathbf{1}_1} - \cdots - \lambda_m \sum_{i=1}^k X^{i+1}\, a_{i\mathbf{1}_m}. \end{aligned} \tag{2.3}$$

Proof. These formulas come out easily from the definition (2.2). ∎

3. Locality of the diffeomorphism, the invariant δ

A map T is called *local* if $\mathrm{Supp}(T(A)) \subset \mathrm{Supp}(A)$ for all $A \in \mathcal{D}_{\underline{\lambda};\mu}$. The well-known Peetre's theorem (cf. [24]) asserts that such a map is a differential operator. Not only is $T(A)$ a differential operator that acts on weighted densities but its coefficients are given by a differential operator as well. The following proposition is adapted from the unary case (see [21]).

Proposition 3.1. *For $k \leq 2$, every $\mathrm{Vect}(M)$-equivariant isomorphism $T : \mathcal{D}^k_{\underline{\lambda};\mu} \to \mathcal{D}^k_{\underline{\rho};\eta}$ is local.*

Proof. Assume that $A \in \mathcal{D}^k_{\underline{\lambda};\mu}$ vanish on an open subset $U \subset M$. We will show that $T(A)$ vanishes on U as well. We have two cases:

(1) The case where $\eta - |\underline{\rho}| \neq 0, 1, 2$. Suppose the contrary, namely $T(A)_{|u} \neq 0$ for some $u \in U$. The principle symbol $\sigma(T(A))$ of $T(A)$ can be expressed as $\oplus_i\, \sigma_i\, (dx)^{\eta-|\underline{\rho}|-p}$, for a certain integer p. We can always choose u such that $\sigma_{i_0|_u} \neq 0$, for a certain i_0. To get the contradiction we will look for a vector field X that satisfies $L^{\underline{\rho};\eta}_X(T(A))_{|u} \neq 0$. Hence the contradiction since

$$L^{\underline{\lambda};\eta}_X(A_{|u}) = 0 \quad \text{and} \quad T\left(L^{\underline{\lambda};\eta}_X(A_{|u})\right) = L^{\underline{\rho};\eta}_X(T(A)_{|u}).$$

To choose X we consider the expression

$$L^{\eta-|\underline{\rho}|-p}_X(\sigma_{i_0}(dx)^{\eta-|\underline{\rho}|-p}) = \left(X(\sigma_{i_0}) + (\eta - |\underline{\rho}| - p)\ \mathrm{div}\ X\sigma_{i_0}\right)(dx)^{\eta-|\underline{\rho}|-p}. \tag{3.1}$$

Since $\eta - |\underline{\rho}| \neq 0, 1, 2$ and $p \leq 2$, we can always choose X such that Eq. (3.1) is not zero. This implies that $L^{\underline{\rho};\eta}_X(T(A))_{|u} \neq 0$ and hence the contradiction.

(2) The case where $\eta - |\underline{\rho}| = 0, 1, 2$. Let $\mathrm{Vect}_U(M)$ be the Lie algebra of smooth vector fields with support in U. Since A vanishes on U, it follows that

$$L_X^{\underline{\lambda};\mu}(A) = 0 \quad \text{for every } X \in \mathrm{Vect}_U(M).$$

Therefore, $L_X^{\underline{\rho};\eta}(T(A)) = 0$ for every $X \in \mathrm{Vect}_U(M)$. This means that the operator $T(A)$ is $\mathrm{Vect}_U(M)$-invariant, as $T(A)$ does not vanish on U. Following [15,18], such an operator can be expressed as follows:

(i) A is the multiplication operator for $\eta - |\underline{\rho}| = 0$,
(ii) $A = \sum_{i,j=1}^m a_{i,j}\{\cdot,\cdot\}$, where $a_{i,j}$ are scalars and $\{\cdot,\cdot\}$ is the Poisson bracket, for $\eta - |\underline{\rho}| = 1$,
(iii) A is a linear combination of operators given by compositions of the de Rham operator and the Poisson bracket for $\eta - |\underline{\rho}| = 2$ and for special values of ρ and $\underline{\eta}$.

Now any operator as in (i)–(iii) is not only $\mathrm{Vect}_U(M)$-invariant but also $\mathrm{Vect}(M)$-invariant. The isomorphism T implies that the operator A is $\mathrm{Vect}(M)$-invariant and is given as in (i)–(iii). This is a contradiction, since if A vanishes on U it must vanish everywhere. ■

Proposition 3.2. *Every linear $\mathfrak{sl}(2)$-equivariant isomorphism $\mathcal{D}_{\underline{\lambda};\mu} \to \mathcal{S}_\delta$ is local.*

Proof. The statement of this proposition, in fact, holds also true for the affine Lie sub-algebra $\mathfrak{a} = \mathrm{Span}\{\partial_x, x\partial_x\}$ of $\mathfrak{sl}(2)$. As $\mathfrak{a}$-modules, the space $\mathcal{D}_{\underline{\lambda};\mu}$ and the space $\mathcal{S}_\delta$ are isomorphic, thanks to Proposition 2.1. Therefore, it is sufficient to prove that any $\mathfrak{a}$-equivariant linear map $\mathcal{F}_{\delta-|\underline{i}|} \to \mathcal{F}_{\delta-|\underline{j}|}$ is local. This has been proved in [21] (Theorem 5.1) using Petree's theorem for $\delta = 0$ but the proof works as well for every δ. ■

For every module $\mathcal{D}^k_{\underline{\lambda};\mu}$, we define its *shift* δ to be

$$\delta := \mu - |\underline{\lambda}|.$$

Proposition 3.3. *A necessary condition for the two modules $\mathcal{D}^k_{\underline{\lambda};\mu}$ and $\mathcal{D}^k_{\underline{\rho};\eta}$ to be isomorphic is to have the same shift.*

Proof. Let $T : \mathcal{D}^k_{\underline{\lambda};\mu} \to \mathcal{D}^k_{\underline{\rho};\eta}$ be a $\mathrm{Vect}(M)$-isomorphism. We shall study the equivariance property with respect to the vector field $X = x\partial_x$ upon taking $\underline{\phi}$ constant. The $\mathrm{Vect}(M)$-equivariance reads as follows:

$$\left(T(L_X^{\underline{\lambda};\mu}A)\right)(\underline{\phi}) = \left(L_X^{\underline{\rho};\eta}T(A)\right)(\underline{\phi}).$$

Consider $A = a_{\underline{0}}\partial_{\underline{0}}$, where $a_{\underline{0}}$ is a smooth function on M, the operator of multiplication. By using Proposition 2.1, we have $L_X^{\underline{\lambda};\mu}A = (L_X^{\mu-|\underline{\lambda}|}\, a_{\underline{0}})\,\partial_{\underline{0}}$. Since T is local (cf. Proposition 3.1) and $\underline{\phi}$ is constant, it follows that (where t_i for $i = 1, \ldots, l$ are smooth functions on M)

$$\left(T(L_X^{\underline{\lambda};\mu}A)\right)(\underline{\phi}) = \sum_{i=0}^{l} t_i \left(L_X^{\mu-|\underline{\lambda}|}a_{\underline{0}}\right)^{(i)} \underline{\phi}. \tag{3.2}$$

On the other hand,

$$\left(L_X^{\underline{\rho};\eta}T(A)\right)(\underline{\phi}) = L_X^{\eta-|\underline{\rho}|}\left(\sum_{i=0}^{l} t_i\, a_{\underline{0}}^{(i)}\right)\underline{\phi}. \tag{3.3}$$

Since T is an isomorphism, the function t_0 is not identically zero. By comparing the coefficient of t_0 in Eqs. (3.2) and (3.3), the result follows. ■

4. The $\mathfrak{sl}(2)$-equivariant symbol calculus

Equivariant symbol calculus was carried out in [5] for the case of binary differential operators. An explicit formula was given for $k = 2$. In this section, we extend the results to m-ary differential operators, exhibiting the symbol map for any k. It should be stressed, however, that the equivariant symbol calculus depends on the embedding of the Lie algebra $\mathfrak{sl}(2) \subset \mathrm{Vect}(M)$. For instance, if we consider the embeddings

$$\mathfrak{sl}(2) \simeq \mathrm{Span}\{\partial_x, \sin(x)\partial_x, \cos(x)\partial_x\} \quad \text{or} \quad \mathfrak{sl}(2) \simeq \mathrm{Span}\{\partial_x, \sinh(x)\partial_x, \cosh(x)\partial_x\},$$

the symbol may not exit, as already pointed out in [3] for the unary case. Throughout this paper, $\mathfrak{sl}(2)$ is realized as in (1.2).

Theorem 4.1. *For all $\delta \notin \{1, \frac{3}{2}, 2, \dots, k\}$, there exits a family of $\mathfrak{sl}(2)$-equivariant maps given by*

$$\sigma^{\alpha}_{\underline{\lambda},\mu} : \mathcal{D}^k_{\underline{\lambda};\mu} \longrightarrow \bigoplus_{j=0}^{k} \mathcal{S}^{(j)}_{\delta-j} \qquad A \mapsto \sum_{r=0}^{k} \sum_{|\underline{i}|=r} \overline{a}_{\underline{i}} (dx)^{\delta-|\underline{i}|}$$

where $\overline{a}_{\underline{i}} = \sum_{|\underline{s}|=|\underline{i}|}^{k} \alpha^{\underline{i}}_{\underline{s}}\, a^{(|\underline{s}|-|\underline{i}|)}_{\underline{i}}$ and the constants $\alpha^{\underline{s}}_{\underline{i}}$ are given by the induction formula (where $|\underline{s}|, |\underline{i}| = 0, \dots, k$):

$$\left(|\underline{s}| - |\underline{i}|\right)\left(2\delta - |\underline{s}| - |\underline{i}| - 1\right)\alpha^{\underline{s}}_{\underline{i}} - \sum_{j=1}^{m} s_j (2\lambda_j + s_j - 1)\, \alpha^{\underline{s}-\mathbf{1}_j}_{\underline{i}} = 0. \tag{4.1}$$

Proof. Proposition 3.2 asserts that the $\mathfrak{sl}(2)$-equivariant symbol map is local, hence it is given by a differential operator. Let us study the invariance property. We shall show that

$$\overline{a}^{X}_{\underline{i}} = L^{\delta-|\underline{i}|}_{X}\, \overline{a}_{\underline{i}} + \text{higher terms } X^{(n)}, \quad n \geq 3.$$

Now if we restrict ourself to $\mathfrak{sl}(2)$ then the second part of the right hand side vanishes and thus we have equivariance. To prove the formula above, we consider

$$\overline{a}_{\underline{i}} = \sum_{|\underline{s}|=|\underline{i}|}^{k} \alpha^{\underline{s}}_{\underline{i}}\, a^{([\underline{s}]-[\underline{i}])}_{\underline{s}}.$$

Upon using Proposition 2.1 we get

$$\begin{aligned} \overline{a}^{X}_{\underline{i}} &= L^{\delta-|\underline{i}|}_{X}\, \overline{a}_{\underline{i}} + X'' \sum_{|\underline{s}|=|\underline{i}|}^{k} \alpha^{\underline{s}}_{\underline{i}} \left(\binom{|\underline{s}-\underline{i}|}{2} + (\delta - |\underline{s}|)|\underline{s}-\underline{i}| \right) a^{(|\underline{s}-\underline{i}|)}_{\underline{s}} \\ &\quad - X'' \sum_{|\underline{s}|=|\underline{i}|-1}^{k-1} \alpha^{\underline{s}}_{\underline{i}} \sum_{j=1}^{m} \left(\lambda_j \binom{s_j+1}{1} + \binom{s_j+1}{2} \right) a^{(|\underline{s}-\underline{i}|+1)}_{\underline{s}} + \text{higher terms in } X^{(n)}. \end{aligned}$$

The coefficient of X'' turns out to be trivial thanks to the formula (4.1) and the induction hypothesis. ■

We have a family of symbol maps $\sigma^{\alpha}_{\underline{\lambda},\mu}$ generated by the entries of the non-singular matrices $[\alpha]_i$, for $i = 1, \dots, k$. Nevertheless, we will prove that once the principal symbol is fixed the symbol map $\sigma^{\alpha}_{\underline{\lambda},\mu}$ is unique.

Proposition 4.2. *For $\delta \notin \{1, \frac{3}{2}, 2, \dots, k\}$, there exists a unique $\mathfrak{sl}(2)$-equivariant symbol map $\sigma_{\underline{\lambda};\mu} : \mathcal{D}_{\underline{\lambda};\mu} \to \mathcal{S}_{\delta}$ such that, for each $A \in \mathcal{D}^k_{\underline{\lambda};\mu}$, the term of highest order of $\sigma_{\lambda;\mu}$ is the symbol map σ^{α} for some matrix α.*

Proof. The proof is similar to [19]. Let assume that there is another $\mathfrak{sl}(2)$-equivariant symbol map $\tilde{\sigma}$. Then, for every integer k, the restriction of the map $\sigma_{\underline{\lambda};\mu} \circ \tilde{\sigma}^{-1}$ to $\mathcal{S}^{(k)}_{\delta-k}$ is of the form

$$a \in \mathcal{S}^{(k)}_{\delta-k} \mapsto (S_0(a), S_1(a), \dots, S_k(a)) \in \bigoplus_{i=0}^{k} \mathcal{S}^{(i)}_{\delta-i}$$

for some $\mathfrak{sl}(2)$-equivariant maps $S_i \in \mathrm{Hom}_{\mathfrak{sl}(2)}(\mathcal{S}^{(k)}_{\delta-k}, \mathcal{S}^{(i)}_{\delta-i})$, where $i = 1, \dots, k$. We have

$$\mathrm{Hom}_{\mathfrak{sl}(2)}(\mathcal{S}^{(k)}_{\delta-k}, \mathcal{S}^{(i)}_{\delta}) = \bigoplus_{\binom{k+m-1}{m-1}\text{times}} \; \bigoplus_{\binom{i+m-1}{m-1}\text{times}} \mathrm{Hom}_{\mathfrak{sl}(2)}(\mathcal{F}_{\delta-k}, \mathcal{F}_{\delta-i}).$$

Following [19] and since $\delta \notin \{1, \frac{3}{2}, 2, \ldots\}$ we have

$$\mathrm{Hom}_{\mathfrak{sl}(2)}(\mathcal{F}_{\delta-k}, \mathcal{F}_{\delta-i}) \simeq \begin{cases} \mathrm{Id} & \text{if } i = k, \\ 0 & \text{if } i \neq k. \end{cases}$$

Therefore, all the maps S_i are zero except S_0 which is given as a multiplication by a non-singular matrix since the maps $\sigma_{\underline{\lambda};\mu}$ and $\tilde{\sigma}$ are isomorphisms. The result follows. ■

We are interested in a class of symbol maps where the principal symbols are given by

$$[\alpha]_i = \mathrm{Id} \quad \text{for } i = 1, \ldots, k.$$

The inverse of the symbol map is the *quantization* map. It is described by the following theorem.

Theorem 4.3. *For all $\delta \notin \{1, 2, \frac{3}{2}, \ldots, k\}$ there exits a family of $\mathfrak{sl}(2)$-equivariant maps given by*

$$Q^{\beta}_{\underline{\lambda},\mu} : \bigoplus_{j=0}^{k} \mathcal{S}^{(j)}_{\delta-j} \longrightarrow \mathcal{D}^{k}_{\underline{\lambda};\mu} \qquad \bigoplus_{r=0}^{k} \bigoplus_{|\underline{i}|=r} a_{\underline{i}} (dx)^{\delta-|\underline{i}|} \mapsto \sum_{r=0}^{k} \sum_{|\underline{i}|=r} \widetilde{a}_{\underline{i}}\, \partial_{\underline{i}} \tag{4.2}$$

where $\widetilde{a}_{\underline{i}} = \sum_{|\underline{s}|=|\underline{i}|}^{k} \beta^{\underline{s}}_{\underline{i}}\, a_{\underline{s}}^{(|\underline{s}-\underline{i}|)}$ and the constants $\beta^{\underline{s}}_{\underline{i}}$ are given by the induction formula (for $|\underline{s}|, |\underline{i}| = 0, \ldots, k$):

$$\left(2\delta - 1 - |\underline{s} - \underline{i}|\right) |\underline{s} - \underline{i}|\, \beta^{\underline{s}}_{\underline{i}} + \sum_{j=1}^{m} (i_j + 1)(2\lambda_j + i_j)\, \beta^{\underline{s}}_{\underline{i}+\mathbf{1}_j} = 0. \tag{4.3}$$

Proof. The proof is similar to that of Theorem 4.1. ■

5. Examples

We provide examples of the quantization map (4.2) for the case of first-order and second-order operators. These expressions will be used to study the $\mathrm{Vect}(M)$-isomorphism problem.

5.1. *The case of first-order m-ary differential operators*

For first-order operators, the quantization map is given as in (4.2), where the constants (4.3) are given by (for $j = 1, \ldots, m$)

$$[\beta]_1 = \mathrm{Id}, \qquad \beta^{\mathbf{1}_j}_{\underline{0}} = \frac{\lambda_j}{1-\delta}. \tag{5.1}$$

5.2. *The case of second-order m-ary differential operators*

For second-order operators, the quantization map is given as in (4.2), where the constants (4.3) are given by (for $i, j = 1, \ldots, m$ and $i \neq j$)

$$[\beta]_2 = \mathrm{Id}, \qquad \beta^{2\mathbf{1}_j}_{\underline{0}} = \frac{\lambda_j(2\lambda_j+1)}{(\delta-2)(2\delta-3)}, \qquad \beta^{\mathbf{1}_i+\mathbf{1}_j}_{\underline{0}} = \frac{2\lambda_i\lambda_j}{(\delta-2)(2\delta-3)}.$$

And

$$\begin{aligned} &\beta^{\mathbf{1}_1+\mathbf{1}_1}_{\mathbf{1}_1} = \frac{2\lambda_1+1}{2-\delta}, \qquad \beta^{\mathbf{1}_1+\mathbf{1}_2}_{\mathbf{1}_1} = \frac{\lambda_2}{2-\delta}, \cdots \qquad \beta^{\mathbf{1}_1+\mathbf{1}_m}_{\mathbf{1}_1} = \frac{\lambda_m}{2-\delta}, \\ &\vdots \\ &\beta^{\mathbf{1}_m+\mathbf{1}_m}_{\mathbf{1}_m} = \frac{2\lambda_m+1}{2-\delta}, \qquad \beta^{\mathbf{1}_m+\mathbf{1}_1}_{\mathbf{1}_m} = \frac{\lambda_1}{2-\delta}, \cdots \qquad \beta^{\mathbf{1}_m+\mathbf{1}_{m-1}}_{\mathbf{1}_m} = \frac{\lambda_{m-1}}{2-\delta}. \end{aligned} \tag{5.2}$$

The other values of $\beta^{\underline{i}}_{\underline{s}}$ vanish.

Remark 5.1. For another approach to the study of the space of m-ary differential operators, see [9].

6. The case of m-ary skew symmetric operators

Consider now the space of m-ary skew symmetric differential operators, $\mathcal{D}_{\wedge^m \lambda;\mu}$. In that case, we deal only with two parameters μ and $\lambda := \lambda_1 = \cdots = \lambda_m$. Let $Q(i,m)$[1] be the number of ways of partitioning i into exactly m distinct positive parts (see [8]). Let us put $R(i,m) = Q(i,m) + Q(i,m-1)$. The quotient module $\mathcal{D}^k_{\wedge^m \lambda;\mu}/\mathcal{D}^{k-1}_{\wedge^m \lambda;\mu}$ can be decomposed into $R(k,m)$-components that transform under coordinate changes as $(\delta-k)$-densities. Denote by $\mathcal{S}^{R(i,m)}_\lambda$ the direct sum $\oplus\mathcal{F}_\lambda$ counted $R(i,m)$-times. The symbol map is as follows:

Theorem 6.1. *For all $\delta \notin \{1, \frac{3}{2}, 2, \ldots, k\}$, there exits a family of $\mathfrak{sl}(2)$-equivariant maps given by*

$$\sigma^\alpha_{\underline{\lambda},\mu} : \mathcal{D}^k_{\wedge^m \lambda;\mu} \longrightarrow \bigoplus_{j=1}^{k} \mathcal{S}^{R(j,m)}_{\delta-j} \qquad A \mapsto \sum_{r=1}^{k} \sum_{\substack{[\underline{i}]=r \\ i_1>\cdots>i_m}} \overline{a}_{\underline{i}} (dx)^{\delta-[\underline{i}]}$$

where $\overline{a}_{\underline{i}} = \sum_{|\underline{s}|=|\underline{i}|}^{k} \alpha^{\underline{i}}_{\underline{s}} a^{(|\underline{s}-\underline{i}|)}_{\underline{i}}$ and the constants $\alpha^{\underline{s}}_{\underline{i}}$ are given by the induction formula (for $|\underline{s}|, |\underline{i}| = 1, \ldots, k$, where $i_1 > \cdots > i_m$ and $s_1 > \cdots > i_m$):

$$|\underline{s}-\underline{i}| \left(2\delta - |\underline{s}-\underline{i}| - 1\right) \alpha^{\underline{s}}_{\underline{i}} - \sum_{j=1}^{m} s_j \,(2\lambda + s_j - 1)\, \alpha^{\underline{s}-\mathbf{1}_j}_{\underline{i}} = 0. \tag{6.1}$$

Proof. The operators that we are dealing with are skew symmetric. Therefore, the components $a_{\underline{0}}$ must be zero and the other components are skew symmetric with respect to the indices. This explains why the sum should be taken over distinct indices. Now the proof is similar to that of Theorem 4.1. ■

Corollary 6.2. *The following $\mathfrak{sl}(2)$-modules are isomorphic*

$$\mathcal{D}^k_{\wedge^2 \lambda;\mu+\lambda} \simeq \bigoplus_{i=0}^{\lfloor \frac{1}{2}(k-1)\rfloor} \mathcal{D}^k_{\lambda;\mu}/\mathcal{D}^i_{\lambda;\mu}.$$

Proof. This follows from Theorem 6.1 and the equivariant quantization map exhibited in [14] for the unary case. ■

Remark 6.3. Skew-symmetric invariant differential operators on weighted densities have been investigated in [12], generalizing the Grozman operator [15] from $\mathcal{D}^3_{\wedge^2 \frac{-2}{3};\frac{5}{3}}$. For a historical account, see [16].

7. The $\mathfrak{sl}(2)$-equivariant quantization, the resonant case

For the sake of completeness, we study the resonant values of $\delta \in \{1, \frac{3}{2}, 2, \frac{5}{2}, \ldots, k\}$. The following result contrasts with the unary case.

Theorem 7.1. (i) *For $\delta = 1$, a $\mathfrak{sl}(2)$-equivariant map $\mathcal{S}_\delta \to \mathcal{D}_{\underline{\lambda};\mu}$ exists only for $\underline{\lambda} = \underline{0}$.*

(ii) *For $\delta = \frac{3}{2}$, a $\mathfrak{sl}(2)$-equivariant map $\mathcal{S}_\delta \to \mathcal{D}_{\underline{\lambda};\mu}$ exists only for $\underline{\lambda} = \underline{0}$ or $\underline{\lambda} = -\frac{1}{2}\mathbf{1}_j$, where $j = 1, \ldots, m$.*

(iii) *For $\delta \in \{2, \frac{5}{2}, 3, \ldots, k\}$ there is no $\mathfrak{sl}(2)$-equivariant map $\mathcal{S}_\delta \to \mathcal{D}_{\underline{\lambda};\mu}$, for any $\underline{\lambda}$ and μ.*

Proof. The proof is based on a mathematical induction. The $\mathfrak{sl}(2)$-equivariance is equivalent to the following linear system (for $|\underline{s}-\underline{i}| \geq 0$ and $|\underline{s}|, |\underline{i}| = 0, \ldots, k$):

$$(2\delta - 1 - |\underline{s}-\underline{i}|)|\underline{s}-\underline{i}|\, \beta^{\underline{s}}_{\underline{i}} + \sum_{j=1}^{m} (i_j+1)(2\lambda_j + i_j)\, \beta^{\underline{s}}_{\underline{i}+\mathbf{1}_j} = 0. \tag{7.1}$$

---

[1] For low values of m, the function $Q(k,m)$ has beautiful expressions. For instance: (1) $Q(i,2) = \lfloor \frac{1}{2}(i-1)\rfloor$, where $\lfloor x\rfloor$ is the floor function defined to be the greatest integer $\leq x$; (2) $Q(i,3) = [\frac{1}{12}(i-3)^2]$, where $[x]$ is the nint function defined to be the closest integer to x with half integers rounded to even numbers, as in $[1.5] = 2$, $[2.5] = 2$.

In the case where $|\underline{s}| \leq k-1$, the system (7.1) is exactly the $\mathfrak{sl}(2)$-equivariance condition for the module $\mathcal{D}^{k-1}_{\underline{\lambda};\mu}$. As we have a filtration of modules

$$\mathcal{D}^{1}_{\underline{\lambda};\mu} \subset \mathcal{D}^{2}_{\underline{\lambda};\mu} \subset \cdots \subset \mathcal{D}^{k-1}_{\underline{\lambda};\mu} \subset \mathcal{D}^{k}_{\underline{\lambda};\mu},$$

we will be dealing with the induction assumption at $k-1$ together with the system (7.1) for $|\underline{s}| = k$.

(i) The case where $\delta = 1$. Let us first study the case where $k = 1$. The $\mathfrak{sl}(2)$-equivariance is equivalent to the system

$$\sum_{j=1}^{m} \beta^{\mathbf{1}_1}_{\mathbf{1}_j} \lambda_j = 0, \ldots, \sum_{j=1}^{m} \beta^{\mathbf{1}_m}_{\mathbf{1}_j} \lambda_j = 0.$$

As the matrix $[\beta]_1$ is non-singular, it follows that $\underline{\lambda}$ must be $\underline{0}$. Suppose that the result holds true at $k-1$. Now, at k we are required to solve the system (7.1) only for $|\underline{s}| = k$ — actually for $|\underline{s}| < k$ solutions are guaranteed by the induction assumption. For this value, the system (7.1) becomes

$$(1-k-|\underline{i}|)(k-|\underline{i}|)\,\beta^{\underline{s}}_{\underline{i}} + \sum_{j=1}^{m}(i_j+1)(2\lambda_j+i_j)\,\beta^{\underline{s}}_{\underline{i}+\mathbf{1}_j} = 0. \tag{7.2}$$

As $1-k-|\underline{i}|$ is never zero for every $|\underline{i}| \leq k-1$, the constant $\beta^{\underline{s}}_{\underline{i}}$ can be expressed in terms of $\beta^{\underline{s}}_{\underline{i}+\mathbf{1}_j}$. This means that every constant $\beta^{\underline{s}}_{\underline{i}}$ can be expressed in terms of the matrix $[\beta]_k$. We have no conditions on that matrix except that it should be non-singular.

(ii) The case where $\delta = \frac{3}{2}$. This value is not a resonant value for the module $\mathcal{D}^{1}_{\underline{\lambda};\mu}$, hence the $\mathfrak{sl}(2)$-equivariant map exists. Here we cannot proceed directly by induction because this value is resonant for the module $\mathcal{D}^{2}_{\underline{\lambda};\mu}$. So we need to prove the result for $k = 2$, then we proceed by induction. The $\mathfrak{sl}(2)$-equivariant is equivalent to the system:

$$-\beta^{\underline{s}}_{\underline{i}} + \sum_{j=1}^{m}(i_j+1)(2\lambda_j+i_j)\,\beta^{\underline{s}}_{\underline{i}+\mathbf{1}_j} = 0 \quad \text{for } |\underline{i}| = 1,$$

$$\sum_{j=1}^{m}(i_j+1)(2\lambda_j+i_j)\,\beta^{\underline{s}}_{\underline{i}+\mathbf{1}_j} = 0 \quad \text{for } |\underline{i}| = 0.$$

By solving this system, we get (for $|\underline{s}|, |\underline{i}| = 2$):

$$\sum_{u=1}^{m}\sum_{v=1}^{m}(i_u-\delta^v_u)(2\lambda_u+i_u-\delta^v_u-1)i_v(2\lambda_v+i_v-1)\beta^{\underline{s}}_{\underline{i}} = 0.$$

As the matrix $[\beta]_2$ is non-singular, the left part $\sum_{u=1}^{m}\sum_{v=1}^{m}(i_u-\delta^v_u)(2\lambda_u+i_u-\delta^v_u-1)i_v(2\lambda_v+i_v-1)$ must be zero. By taking $\underline{i} = 2\mathbf{1}_r$ (for $r = 1, \ldots, m$) we obtain the system

$$4\lambda_r(2\lambda_r+1) = 0. \tag{7.3}$$

By taking $\underline{i} = \mathbf{1}_p + \mathbf{1}_q$ (for $p, q = 1, \ldots, m$), we obtain the system

$$\lambda_p\,\lambda_q = 0. \tag{7.4}$$

The system (7.3) and (7.4) admits roots, given as stated in the theorem. Suppose that the result holds true at $k-1$. As explained in Part (i), we will deal only with the system (7.1) for $|\underline{s}| = k$. For this value, the system (7.1) becomes

$$(2-k-|\underline{i}|)(k-|\underline{i}|)\,\beta^{\underline{s}}_{\underline{i}} + \sum_{j=1}^{m}(i_j+1)(2\lambda_j+i_j)\,\beta^{\underline{s}}_{\underline{i}+\mathbf{1}_j} = 0. \tag{7.5}$$

As the quantity $(2-k-|\underline{i}|)$ is never zero, for every $|\underline{i}| \leq k-1$ and $k > 2$, the constant $\beta^{\underline{s}}_{\underline{i}}$ can be expressed in terms of $\beta^{\underline{s}}_{\underline{i}+\mathbf{1}_j}$. This means that every constant $\beta^{\underline{s}}_{\underline{i}}$ can be expressed in terms of the matrix $[\beta]_k$. We have no conditions on that matrix except being non-singular. The result follows upon taking the restriction to the submodule $\mathcal{D}^{k-1}_{\underline{\lambda},\mu}$.

(iii) The case where $\delta \in \{2, \frac{5}{2}, \ldots, k\}$. Let us start by studying the case where δ is an integer. The proof can be obtained for $k = 2$ but we here omit the details. Suppose that the result is true at $k-1$. The inclusion $\mathcal{D}^{k-1}_{\underline{\lambda},\mu} \subset \mathcal{D}^{k}_{\underline{\lambda};\mu}$ implies that the $\mathfrak{sl}(2)$-equivariant map does not exist for $\delta = 2, \ldots, k-1$. Let us prove the result for $\delta = k$. For this value the system (7.1), for $|\underline{s}| = k$, becomes (where $|\underline{i}| = k$):

$$\sum_{j=1}^{m} i_j\,(2\lambda_j + i_j - 1)\,\beta_{\underline{i}}^{\underline{s}} = 0. \tag{7.6}$$

For $i = 2\mathbf{1}_1 + (k-2)\mathbf{1}_2$, the system (7.6) can be rewritten as

$$[\beta]_k \times \begin{bmatrix} 0 \\ \vdots \\ 2(2\lambda_1+1) \\ (k-2)(2\lambda_2+k-3) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

For $i = \mathbf{1}_1 + (k-1)\mathbf{1}_2$, the system (7.6) can be rewritten as

$$[\beta]_k \times \begin{bmatrix} 0 \\ \vdots \\ 2\lambda_1 \\ (k-1)(2\lambda_2+k-2) \\ 0 \\ \vdots \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

As the matrix $[\beta]_k$ is not singular, it follows that $\lambda_1 = -\frac{1}{2}$ and $\lambda_1 = 0$ which is absurd.

Let us study the case where $\delta = \frac{2l-1}{2}$ for $l = 3, \ldots, k$. Here the computation can be checked for $k = 3$ but we omit the details. Suppose that the result is true at $k-1$. The inclusion $\mathcal{D}^{k-1}_{\underline{\lambda},\mu} \subset \mathcal{D}^{k}_{\underline{\lambda};\mu}$ implies that the $\mathfrak{sl}(2)$-equivariant map does not exist for $\delta = \frac{2l-1}{2}$, where $l = 3, \ldots, k-1$. Let us prove the result for $\delta = \frac{2k-1}{2}$. This value is, actually, not a resonant value for the module $\mathcal{D}^{k-1}_{\underline{\lambda},\mu}$. Therefore, the system (7.1) admits a solution for every $|\underline{s}| < k$. Let us study this system for $|\underline{s}| = k$. By solving the system (7.1) for $|\underline{i}| = k-1$ we get

$$\beta_{\underline{i}}^{\underline{s}} = \sum_{j=1}^{m} (i_j+1)(2\lambda_j + i_j)\,\beta_{\underline{i}+\mathbf{1}_j}^{\underline{s}}. \tag{7.7}$$

Now for $|\underline{i}| = k-2$, the system (7.1) becomes

$$\sum_{j=1}^{m} i_j\,(2\lambda_j + i_j - 1)\beta_{\underline{i}}^{\underline{s}} = 0. \tag{7.8}$$

Upon substituting Eq. (7.7) into Eq. (7.8) we get (for $|\underline{s}|, |\underline{i}| = k$)

$$\sum_{u=1}^{m}\sum_{v=1}^{m} (i_u - \delta_u^v)(2\lambda_u + i_u - \delta_u^v - 1)i_v(2\lambda_v + i_v - 1)\beta_{\underline{i}}^{\underline{s}} = 0.$$

As the matrix $[\beta]_k$ is non-singular, the left part $\sum_{u=1}^{m}\sum_{v=1}^{m}(i_u - \delta_u^v)(2\lambda_u + i_u - \delta_u^v - 1)i_v(2\lambda_v + i_v - 1)$ must be zero. By taking $\underline{i} = k\mathbf{1}_r$ (for $r = 1, \ldots, m$) we obtain the system

$$(1-k)(2\lambda_r + k - 2)(2\lambda_r + k - 1) = 0. \tag{7.9}$$

By taking $\underline{i} = (k-1)\mathbf{1}_p + \mathbf{1}_q$ (for $p, q = 1, \ldots, m,$ and $p \neq q$) we obtain the system

$$(k-1)(2\lambda_p + k - 2)\,((k-2)(2\lambda_p + k - 3) + 4\lambda_q) = 0. \tag{7.10}$$

By taking $\underline{i} = (k-2)\mathbf{1}_p + \mathbf{1}_q + \mathbf{1}_r$ (for $p, q, r = 1, \ldots, m,$ and p, q, r are distinct) we obtain the system

$$(k-2)[(2\lambda_p + k - 3)((k-3)(2\lambda_p + k - 4) + 4\,\lambda_q + 4\,\lambda_r) + 8\lambda_q\lambda_r] = 0. \tag{7.11}$$

We distinguish two cases:

(1) If $\lambda_i = \frac{2-k}{2}$ for all $i = 1, \ldots, m$. By substituting λ_i in Eq. (7.11) we get

$$\frac{1}{k}(2k-3)(k-2).$$

This outcome is never zero for all $k \geq 3$.

(2) If there exists i_0 such that $(2\lambda_{i_0} + k - 2) \neq 0$, Eq. (7.9) implies that $\lambda_{i_0} = \frac{1-k}{2}$. Now Eq. (7.10) implies that $\lambda_j = \frac{k-2}{2}$ for all $j \neq i_0$. By substituting in Eq. (7.11) we get

$$-2(k-2)(k-1)(2k-3).$$

This last outcome is never zero for all $k \geq 3$.

Thus, the system (7.1) has no solutions and a fortiori there is no $\mathfrak{sl}(2)$-equivariant quantization map. ■

8. A remark on Vect(M)-equivariant quantization

The $\mathfrak{sl}(2)$-equivariant quantization map is not unique, generated by the entries of the matrices $[\beta]_i$, where $i = 1, \ldots, k$. We can ask whether there exists an appropriate principal symbol for which the equivariant quantization maps turn into Vect(M)-equivariant ones.

Theorem 8.1. *For $\delta \notin \{1, \frac{3}{2}, \ldots, k\}$, there exists a principal symbol for which the corresponding quantization map is* Vect(M)*-equivariant only in the following cases:*

1. *For $k = 1$.*
2. *For $k = 2$ but $\underline{\lambda} = \underline{0}$ or $\underline{\lambda} = (1-\delta)\mathbf{1}_j$ for $j = 1, \ldots, m$.*

Proof. We will first prove the result for $k = 1, 2$ and 3. For $k = 3$, we will prove that no such principal symbol exists. As we have a filtration of modules

$$\mathcal{D}^2_{\underline{\lambda},\mu} \subset \mathcal{D}^3_{\underline{\lambda},\mu} \subset \cdots \subset \mathcal{D}^k_{\underline{\lambda};\mu},$$

the result holds for any $k > 3$ upon taking the restriction to the module $\mathcal{D}^3_{\underline{\lambda},\mu}$ and applying the result.

For $k = 1$, the Vect(M)-equivariance is given by the system (4.3) (for $|\underline{s}|, |\underline{i}| = 0, 1$). Upon solving this system we get

$$\beta_{\underline{0}}^{\mathbf{1}_j} = \sum_{s=1}^{m} \frac{\lambda_s}{1-\delta}\beta_{\mathbf{1}_s}^{\mathbf{1}_j} \quad \text{for } j = 1, \ldots, m.$$

There are no more conditions on the constants $\beta_{\mathbf{1}_s}^{\mathbf{1}_j}$ except that $\mathrm{Det}[\beta]_1 \neq 0$. We can take, for instance, $[\beta]_1 = \mathrm{Id}$, and therefore the corresponding quantization map is certainly Vect(M)-equivariant.

For $k = 2$, the Vect(M)-equivariance is given by the system (4.3) (for $\underline{s}, \underline{i} = 0, 1, 2$) together with the following system (for $\underline{u} = k$):

$$\sum_{j=1}^{m} \lambda_j \beta_{\mathbf{1}_j + \mathbf{1}_j}^{\underline{u}} + (\delta - 2)\beta_{\underline{0}}^{\underline{u}} = 0. \tag{8.1}$$

By solving the system (4.3), we get

$$\beta_{\underline{0}}^{\underline{u}} = \sum_{j=1}^{m} \frac{\lambda_j(2\lambda_j + 1)}{(\delta-2)(2\delta-3)}\beta_{\mathbf{1}_j+\mathbf{1}_j}^{\underline{u}} + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \frac{2\lambda_j\lambda_i}{(\delta-2)(2\delta-3)}\beta_{\mathbf{1}_i+\mathbf{1}_j}^{\underline{u}}.$$

By substituting into Eq. (8.1) we get a new system (for $|\underline{u}| = m$):

$$\sum_{j=1}^{m} \lambda_j(\lambda_j + \delta - 1)\beta^{\underline{u}}_{\mathbf{2}\mathbf{1}_j} + \sum_{\substack{i,j=1\\ i\neq j}}^{m} \lambda_i\lambda_j\beta^{\underline{u}}_{\mathbf{1}_i+\mathbf{1}_j} = 0.$$

This system admits a solution for which the matrix $[\beta]_2$ is non-singular if and only if the weights $\underline{\lambda}$ are given as in Theorem 8.1.

For $k = 3$, we proceed as above; two systems will be obtained that we solve for the particular values of the weight $\underline{\lambda}$. We omit details here but the proof is just a direct computation. ■

9. Conjugation of m-ary differential operators

First, we define a natural Vect(M)-isomorphism on the modules $\mathcal{D}_{\underline{\lambda};\mu}$ by just permuting arguments. Consider the map $\mathrm{per}_{i,j}$ that interchanges an element at the ith position with an element at the jth position. This map induces an isomorphism (for $i, j = 1, \ldots, m$ and $i \neq j$)

$$\mathcal{D}_{\underline{\lambda};\mu} \to \mathcal{D}_{\mathrm{per}_{i,j}(\underline{\lambda});\mu} \qquad A \mapsto A \circ \mathrm{per}_{i,j}.$$

Now, we will define the notion of conjugation for m-ary differential operators. For $M = \mathbb{R}$, we consider compactly-supported densities.

Upon using successive integration by parts, we get

$$\int_M A(\varphi_1, \ldots, \varphi_m)\phi = \int_M \varphi_1 A^*(\varphi_2, \ldots, \varphi_m, \phi),$$

where $A^*(\varphi_2, \ldots, \varphi_m, \phi) = \sum_{\underline{i}} (-1)^{i_1} \partial_{i_1} (a_{\underline{i}}\, \partial_{i_2}(\varphi_2) \cdots \partial_{i_m}(\varphi_m)\, \phi)$. Therefore, the map $*$ induces a Vect(M)-isomorphism

$$\mathcal{D}_{\lambda_1,\ldots,\lambda_m;\mu} \xrightarrow{\simeq} \mathcal{D}_{\lambda_2,\ldots,\lambda_m,1-\mu;1-\lambda_1} \qquad A \mapsto A^*.$$

The following definition is adapted from the unary case [14].

Definition 9.1. A module $\mathcal{D}_{\underline{\lambda};\mu}$ is said to be singular if either it is only isomorphic to itself, or it is isomorphic to any another module $\mathcal{D}_{\underline{\rho};\eta}$ only through compositions of conjugations and permutations.

10. Classification of the modules $\mathcal{D}^2_{\underline{\lambda};\mu}$

In this section we tackle the isomorphism problem. We study only the case of second-order differential operators. The case where $k > 2$ seems to be more intricate. We need the following

Proposition 10.1. *Every isomorphism $T : \mathcal{D}^k_{\underline{\lambda};\mu} \to \mathcal{D}^k_{\underline{\rho};\varrho}$ is block diagonal in terms of the $\mathfrak{sl}(2)$-equivariant symbols. Namely, the map $\sigma^{\mathrm{Id}}_{\underline{\lambda};\mu} \circ T \circ Q^{\mathrm{Id}}_{\underline{\rho};\varrho} : \bigoplus_{i=0}^k \mathcal{S}^{(i)}_{\delta-i} \to \bigoplus_{i=0}^k \mathcal{S}^{(i)}_{\delta-i}$ is given by (where $\tau^{\underline{i}}_{\underline{s}}$ are constants)*

$$\bigoplus_{|\underline{i}|=0}^{k} a_{\underline{i}} \mapsto \bigoplus_{|\underline{s}|=0}^{k} \sum_{|\underline{i}|=|\underline{s}|} \tau^{\underline{i}}_{\underline{s}}\, a_{\underline{i}}, \tag{10.1}$$

and $[\tau]_i$ are non-singular matrices for $i = 0, 1, \ldots, k$.

Proof. As T is Vect(M)-equivariant, it follows that the composition

$$\mathcal{D}^k_{\underline{\lambda};\mu} \xrightarrow{T} \mathcal{D}^k_{\underline{\rho};\eta} \xrightarrow{\sigma^{\mathrm{Id}}_{\underline{\rho};\eta}} \bigoplus_{i=0}^{k} \mathcal{S}^{(i)}_{\delta-i} \tag{10.2}$$

is $\mathfrak{sl}(2)$-equivariant. Therefore, it coincides with the symbol map $\sigma^{\tau}_{\underline{\lambda};\mu}$ for some τ. Namely, $\sigma^{\mathrm{Id}}_{\underline{\rho};\eta}\circ T=\sigma^{\tau}_{\underline{\lambda};\mu}$. It follows that

$$\sigma^{\mathrm{Id}}_{\underline{\rho};\eta}\circ T\circ Q^{\mathrm{Id}}_{\underline{\lambda};\mu}=\sigma^{\tau}_{\underline{\lambda};\mu}\circ Q^{\mathrm{Id}}_{\underline{\lambda};\mu}.$$

Now, it is a matter of a direct computation to prove that $\sigma^{\tau}_{\underline{\lambda};\mu}\circ Q^{\mathrm{Id}}_{\underline{\lambda};\mu}$ is given as (10.1). ■

10.1. The generic case

We start by studying the case where $\delta\neq 1,\frac{3}{2},2$.

Theorem 10.2. (i) *For $\delta\neq 1$, all modules $\mathcal{D}^1_{\underline{\lambda};\mu}$ are isomorphic provided they have the same shift δ.*

(ii) *For $\delta\neq 1,\frac{3}{2},2$, all modules $\mathcal{D}^2_{\underline{\lambda};\mu}$ are isomorphic provided they have the same shift δ; however, the modules*

$$\mathcal{D}^2_{\underline{0};\delta}\simeq\mathcal{D}^2_{(1-\delta)\mathbf{1}_1;1}\simeq\cdots\simeq\mathcal{D}^2_{(1-\delta)\mathbf{1}_m;1}$$

are singular.

Proof. By virtue of Proposition 10.1 we deal with the map

$$\sigma^{\mathrm{Id}}_{\underline{\lambda};\mu}\circ T\circ Q^{\mathrm{Id}}_{\underline{\rho};\varrho}:\bigoplus_{i=0}^{k}\mathcal{S}^{(i)}_{\delta-i}\to\bigoplus_{i=0}^{k}\mathcal{S}^{(i)}_{\delta-i}\qquad\bigoplus_{|\underline{i}|=0}^{k}a_{\underline{i}}\mapsto\bigoplus_{|\underline{s}|=0}^{k}\sum_{|\underline{i}|=|\underline{s}|}\tau^{\underline{i}}_{\underline{s}}\,a_{\underline{i}}.$$

We are required to exhibit the coefficients $(\tau^{\underline{s}}_{\underline{i}})$. For this matter, we need to compute the action

$$\sigma^{\mathrm{Id}}_{\underline{\lambda};\mu}\circ L^{\underline{\lambda},\mu}_X\circ Q^{\mathrm{Id}}_{\underline{\rho};\varrho}.\tag{10.3}$$

For Part (i), a direct computation shows that the action on $\mathcal{S}^{(1)}_{\delta-1}\oplus\mathcal{S}_\delta$ reads as follows

$$\begin{aligned}\overline{a}^X_{\underline{i}}&=L^{\delta-1}_X a_{\underline{i}}\quad\text{for }|\underline{i}|=1,\\ \overline{a}^X_{\underline{0}}&=L^{\delta}_X a_{\underline{0}}.\end{aligned}$$

Therefore, the modules $\mathcal{D}^1_{\underline{\lambda};\mu}$ and $\mathcal{S}^{(1)}_{\delta-1}\oplus\mathcal{S}_\delta$ are isomorphic to each other. We can choose the parameters $\tau^{\underline{s}}_{\underline{i}}$ as

$$[\tau]_1=\mathrm{Id}\quad\text{and}\quad\tau_{\underline{0}}=1.$$

For Part (ii), a direct computation shows that the action (10.3) on $\mathcal{S}^{(2)}_{\delta-2}\oplus\mathcal{S}^{(1)}_{\delta-1}\oplus\mathcal{S}_\delta$ reads as follows:

$$\begin{aligned}\overline{a}^X_{\underline{i}}&=L^{\delta-2}_X a_{\underline{i}}\quad\text{for }|\underline{i}|=2,\\ \overline{a}^X_{\underline{i}}&=L^{\delta-1}_X a_{\underline{i}}\quad\text{for }|\underline{i}|=1,\\ \overline{a}^X_{\underline{0}}&=L^{\delta}_X a_{\underline{0}}+\sum_{|\underline{i}|=2}\alpha_{\underline{i}}\,X'''\,a_{\underline{i}},\end{aligned}\tag{10.4}$$

where (for $s,t=1,\dots,m$ and $s\neq t$):

$$\alpha_{2\mathbf{1}_s}=2\lambda_s\frac{1-\delta-\lambda_s}{2\delta-3},\qquad\alpha_{\mathbf{1}_s+\mathbf{1}_t}=-2\frac{\lambda_s\lambda_t}{2\delta-3}.$$

The action (10.4) cannot be the action (2.2) because the 1-cocycle

$$\mathrm{Vect}(M)\to\mathcal{D}_{\theta;\theta+2}\qquad(X,\phi)\mapsto X'''\phi,$$

is not trivial for $\theta\neq-\frac{1}{2}$ (cf. [4,13]). We define the column matrix (of $\binom{m+1}{m-1}$-entries) by

$$\alpha(\underline{\lambda},\mu)=\begin{bmatrix}\alpha_{2\mathbf{1}_1}(\underline{\lambda},\mu)\\ \alpha_{\mathbf{1}_1+\mathbf{1}_2}(\underline{\lambda},\mu)\\ \vdots\\ \alpha_{2\mathbf{1}_m}(\underline{\lambda},\mu)\end{bmatrix}.$$

The existence of the Vect(M)-isomorphism is equivalent to solving the linear system

$$[\tau]_2 \cdot \alpha(\underline{\lambda}, \mu) = \alpha^t(\underline{\rho}, \varrho) \quad \text{and} \quad \mathrm{Det}[\tau]_2 \neq 0. \tag{10.5}$$

We distinguish two cases:

(1) If all entries of the column matrix $\alpha(\underline{\lambda}, \mu)$ are zero so are the entries of the row matrix $c(\underline{\rho}, \varrho)$, as $\mathrm{Det}[\tau]_2 \neq 0$. Now, the roots of the equation $\alpha(\underline{\lambda}, \mu) = 0$ are

$$\underline{\lambda} = \underline{0} \quad \text{or} \quad (1-\delta)\mathbf{1}_j \quad \text{for } j = 1, \ldots, m. \tag{10.6}$$

Besides, the values of $\underline{\rho}$ must also be in the form (10.6). Therefore, the corresponding isomorphism is the composition of permutations and conjugations. Thus, the modules (where $\underline{\lambda}$ is as in (10.6))

$$\mathcal{D}^2_{\underline{\lambda};\mu}$$

are singular.

(2) If the column matrix $\alpha(\underline{\lambda}, \mu)$ is not identically zero neither is the column matrix $\alpha(\underline{\rho}, \varrho)$. Whatever the weights $\underline{\lambda}$ and $\underline{\rho}$ are, the constant $\tau^{\underline{i}}_{\underline{s}}$ can be chosen such that the conditions (10.5) are satisfied. Thus, all modules $\mathcal{D}^2_{\underline{\lambda};\mu}$ are isomorphic to each other. ■

Remark 10.3. The non-uniqueness of the isomorphism T is also a characteristic feature of the m-ary case.

10.2. The resonant case; the case of binary operators

Throughout this section we deal with $m = 2$; thus for instance $\underline{\lambda}$ stands for (λ_1, λ_2). We shall study the case when $\delta = 1, \frac{3}{2}, 2$. The quantization map exists only for some particular values of $\underline{\lambda}$. Hence the techniques used in the previous section do not work. Here, we proceed explicitly.

Theorem 10.4. *For $\delta = 1$, all modules $\mathcal{D}^1_{\underline{\lambda};1+[\underline{\lambda}]}$ are isomorphic. However, we have one exceptional module*

$$\mathcal{D}^1_{\underline{0};1}.$$

Proof. We establish an isomorphism between the modules $\mathcal{D}^1_{\underline{\lambda};[\underline{\lambda}]+1} \to \mathcal{D}^1_{\underline{\rho};[\underline{\rho}]+1}$ as follows

(1) If $\rho_2 = 0$, then take T as

$$(a_{\underline{i}}, a_{\underline{0}}) \mapsto \left(\sum_{[\underline{s}]=1} \tau^{\underline{s}}_{\underline{i}} a_{\underline{s}}, a_{\underline{0}} \right) \quad \text{for } [\underline{i}] = 1,$$

where the matrix $[\tau]_1 = \begin{bmatrix} \frac{\lambda_1}{\rho_1} & \frac{\lambda_2}{\rho_2} \\ c_1 & c_2 \end{bmatrix}$. Whatever the values λ_1 and λ_2 can take, the constants c_1 and c_2 can be chosen in a way such that $\mathrm{Det}[\tau]_1 \neq 0$.

(2) If $\rho_2, \rho_1 \neq 0$, then take T as

$$(a_{\underline{i}}, a_{\underline{0}}) \mapsto \left(\sum_{[\underline{s}]=1} \tau^{\underline{s}}_{\underline{i}} a_{\underline{s}}, a_{\underline{0}} \right) \quad \text{for } [\underline{i}] = 1,$$

where the matrix $[\tau]_1 = \begin{bmatrix} \frac{\lambda_1}{\rho_1} & 0 \\ 0 & \frac{\lambda_2}{\rho_2} \end{bmatrix}$. We point out that $\lambda_1, \lambda_2 \neq 0$; otherwise, we go back to Part 1.

Suppose now that $\mathcal{D}^1_{\underline{\lambda};[\underline{\lambda}]+1}$ is isomorphic to $\mathcal{D}^1_{\underline{0};1}$. Therefore, the composition map

$$\mathcal{D}^1_{\underline{\lambda};[\underline{\lambda}]+1} \to \mathcal{D}^1_{\underline{0};1} \to \mathcal{F}^{(1)}_{\delta-1} \oplus \mathcal{F}_\delta,$$

is a $\mathfrak{sl}(2)$-equivariant quantization map in contradiction with Theorem 7.1. ■

Theorem 10.5. *For $\delta = 2$, we have two classes of binary differential operators*

$$\mathcal{D}^2_{s\mathbf{1}_2;2+s} \simeq \mathcal{D}^2_{s\mathbf{1}_1;2+s} \quad \textit{and} \quad \mathcal{D}^2_{\underline{\lambda};2+[\underline{\lambda}]}\,(\lambda_1,\lambda_2 \neq 0).$$

However, we have two exceptional modules:

$$\mathcal{D}^2_{-\mathbf{1}_1;1} \simeq \mathcal{D}^2_{-\mathbf{1}_2;1} \simeq \mathcal{D}^2_{\underline{0};2} \quad \textit{and} \quad \mathcal{D}^2_{-\frac{1}{2}\mathbf{1}_1;\frac{3}{2}} \simeq \mathcal{D}^2_{-\frac{1}{2}\mathbf{1}_2;\frac{3}{2}} \simeq \mathcal{D}^2_{\underline{-1/2};1} \quad (\textit{conjugations}).$$

(iii) *For $\delta = 1$, all modules are isomorphic. However, we have one singular module*

$$\mathcal{D}^2_{\underline{0};1}.$$

(iv) *For $\delta = 3/2$, all modules are isomorphic. However we have one singular module*

$$\mathcal{D}^2_{-\frac{1}{2}\mathbf{1}_1;1} \simeq \mathcal{D}^2_{-\frac{1}{2}\mathbf{1}_2;1} \simeq \mathcal{D}^2_{\underline{0};\frac{3}{2}} \quad (\textit{conjugations}).$$

Proof. By using Proposition 3.1, every isomorphism $T : \mathcal{D}^2_{\underline{\lambda};\mu} \to \mathcal{D}^2_{\underline{\rho};\eta}$ is local. Therefore, the map T retains the following general form (where $[i] = 2$ and $[j] = 1$)

$$(a_{\underline{i}}, a_{\underline{j}}, a_{\underline{0}}) \mapsto \left(\sum_{[\underline{s}]=2} \tau^{\underline{s}}_{\underline{i}}\, a_{\underline{s}},\ \sum_{[\underline{s}]=2} \tau^{\underline{s}}_{\underline{j}}\, a'_{\underline{s}} + \sum_{[\underline{s}]=1} \tau^{\underline{s}}_{\underline{j}}\, a_{\underline{s}},\ \sum_{[\underline{s}]=2} \tau^{\underline{s}}_{\underline{0}}\, a''_{\underline{s}} + \sum_{[\underline{s}]=1} \tau^{\underline{s}}_{\underline{0}}\, a'_{\underline{s}} + \tau_{\underline{0}}\, a_{\underline{0}} \right).$$

A long and tedious computation proves that the Vect($\mathbb{R}$)-equivariant property is equivalent to the following system of fourteen equations:

$$\lambda_1\, \tau^{\mathbf{1}_2}_{\underline{0}} + \lambda_2\, \tau^{\mathbf{1}_1}_{\underline{0}} - (\delta - 2)\tau^{\mathbf{1}_1+\mathbf{1}_2}_{\underline{0}} - \sum_{j=1}^{2} \rho_j\, \tau^{\mathbf{1}_1+\mathbf{1}_2}_{\mathbf{2}\mathbf{1}_j} = 0$$

$$(1 + 2\lambda_s)\, \tau^{\mathbf{1}_s}_{\underline{0}} - (2\delta - 3)\tau^{\mathbf{2}\mathbf{1}_s}_{\underline{0}} - \sum_{j=1}^{2} \rho_j\, \tau^{\mathbf{2}\mathbf{1}_s}_{\mathbf{1}_j} = 0 \quad \text{for } s = 1, 2$$

$$\lambda_1\, \tau^{\mathbf{1}_2}_{\underline{0}} + \lambda_2\, \tau^{\mathbf{1}_1}_{\underline{0}} - (2\delta - 3)\tau^{\mathbf{1}_1+\mathbf{1}_2}_{\underline{0}} - \sum_{j=1}^{2} \rho_j\, \tau^{\mathbf{1}_1+\mathbf{1}_2}_{\mathbf{1}_i} = 0$$

$$\lambda_s - (\delta - 1)\, \tau^{\mathbf{1}_s}_{\underline{0}} - \sum_{j=1}^{2} \rho_j\, \tau^{\mathbf{1}_s}_{\mathbf{1}_j} = 0 \quad \text{for } s = 1, 2$$

$$(1 + 2\lambda_s)\, \tau^{\mathbf{1}_s}_{\mathbf{1}_j} - (\delta - 2)\tau^{\mathbf{2}\mathbf{1}_s}_{\mathbf{1}_j} - (1 + 2\rho_s)\, \tau^{\mathbf{2}\mathbf{1}_s}_{\mathbf{2}\mathbf{1}_j} - \rho_j\, \tau^{\mathbf{2}\mathbf{1}_s}_{\mathbf{1}_1+\mathbf{1}_2} = 0 \quad \text{for } s, j = 1, 2$$

$$\sum_{i=1}^{2} \lambda_j\, \tau^{\mathbf{1}_j}_{\mathbf{1}_s} - (\delta - 2)\, \tau^{\mathbf{1}_1+\mathbf{1}_2}_{\mathbf{1}_s} - (1 + 2\rho_s)\, \tau^{\mathbf{1}_1+\mathbf{1}_2}_{\mathbf{2}\mathbf{1}_s} - \rho_{s+1}\, \tau^{\mathbf{1}_1+\mathbf{1}_2}_{\mathbf{1}_1+\mathbf{1}_2} = 0 \quad \text{for } s = 1, 2$$

$$\lambda_s + (1 + 2\lambda_s)\, \tau^{\mathbf{1}_s}_{\underline{0}} - (\delta - 2)\tau^{\mathbf{2}\mathbf{1}_s}_{\underline{0}} - \sum_{i=1}^{2} \rho_i\, \tau^{\mathbf{2}\mathbf{1}_s}_{\mathbf{2}\mathbf{1}_i} = 0 \quad \text{for } s = 1, 2.$$

We give the details of the computation only for $\delta = \frac{3}{2}$. Here we distinguish also many cases:

(1) If $\rho_1, \rho_2 \neq 0$, then

$$[\tau]_1 = \mathrm{Id} \quad \text{and} \quad [\tau]_2 = \begin{bmatrix} 1 & \dfrac{(\lambda_1 - \rho_1)(1 + 2\lambda_1 + 2\rho_1)}{2\rho_1\rho_2} & 0 \\ 0 & \dfrac{(\lambda_2 - \rho_2)(1 + 2\lambda_2 + 2\rho_2)}{2\rho_1\rho_2} & 1 \\ 0 & \dfrac{\lambda_1\lambda_2}{\rho_1\rho_2} & 0 \end{bmatrix},$$

together with (where $s, j = 1, \ldots, m$)

$$\tau_{\underline{0}}^{\mathbf{2}\mathbf{1}_s} = (6+8\lambda_s)(\rho_s-\lambda_s), \qquad \tau_{\underline{0}}^{\mathbf{1}_1+\mathbf{1}_2} = 4(\lambda_2\rho_1+\lambda_1(\rho_2-2\lambda_2)),$$
$$\tau_{\underline{0}}^{\mathbf{1}_s} = 2(\lambda_s-\rho_s), \qquad \tau_{\mathbf{1}_s}^{\mathbf{1}_1+\mathbf{1}_2} = 2\lambda_{s+1}(-1+\frac{\lambda_s}{\rho_s}),$$
$$\tau_{\mathbf{1}_j}^{\mathbf{2}\mathbf{1}_s} = \frac{(\lambda_s-\rho_s)(1+2\lambda_s-2\rho_s)}{\rho_j}.$$

(2) If $\rho_1 = 0$ but $\rho_2 \neq 0$, then

$$[\tau]_1 = \mathrm{Id} \quad \text{and} \quad [\tau]_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & x & y \\ \dfrac{\lambda_1(1+2\lambda_1)}{\rho_2(1+2\rho_2)} & \dfrac{2\lambda_1\lambda_2}{\rho_2(1+2\rho_2)} & \dfrac{\lambda_2(1+2\lambda_2)}{\rho_2(1+2\rho_2)} \end{bmatrix},$$

where the constants x and y can be chosen in a way such that $\mathrm{Det}[\tau]_2 \neq 0$. The other constants are given by

$$\tau_{\underline{0}}^{\mathbf{2}\mathbf{1}_2} = -\frac{4(\lambda_2-\rho_2)(1+\lambda_2+2\rho_2+4\lambda_2\rho_2)}{(1+2\rho_2)}, \qquad \tau_{\underline{0}}^{\mathbf{2}\mathbf{1}_1} = -\frac{4\lambda_1(1+\lambda_1+3\rho_2+4\lambda_1\rho_2)}{(1+2\rho_2)},$$
$$\tau_{\underline{0}}^{\mathbf{1}_2} = 2(\lambda_2-\rho_2), \qquad \tau_{\underline{0}}^{\mathbf{1}_1} = 2\lambda_1,$$
$$\tau_{\mathbf{1}_1}^{\mathbf{2}\mathbf{1}_2} = 2\rho_2 y, \qquad \tau_{\mathbf{1}_1}^{\mathbf{1}_1+\mathbf{1}_2} = 2(-\lambda_2+x\,\rho_2),$$
$$\tau_{\mathbf{1}_1}^{\mathbf{2}\mathbf{1}_2} = -4\lambda_1, \qquad \tau_{\mathbf{1}_2}^{\mathbf{2}\mathbf{1}_2} = \frac{2(1+2\lambda_2)(\lambda_2-\rho_2)}{\rho_2},$$
$$\tau_{\mathbf{1}_2}^{\mathbf{1}_1+\mathbf{1}_2} = \lambda_1\left(-2+\frac{4\lambda_2}{\rho_2}\right), \qquad \tau_{\mathbf{1}_1}^{\mathbf{2}\mathbf{1}_2} = \frac{2\lambda_1(1+2\lambda_1)}{\rho_2}.$$

Theorem 8.1 asserts that $\mathcal{D}^2_{\underline{\lambda};[\underline{\lambda}]+\frac{3}{2}}$, for $\underline{\lambda} \neq \underline{0}, -\frac{1}{2}\mathbf{1}_1, -\frac{1}{2}\mathbf{1}_2$, is not isomorphic to $\mathcal{D}^2_{\underline{0};\frac{3}{2}} \simeq \mathcal{D}^2_{-\frac{1}{2}\mathbf{1}_1;1} \simeq \mathcal{D}^2_{-\frac{1}{2}\mathbf{1}_2;1}$. ■

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